Automated determination of isoptics with dynamic geometry

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Abstract

We present two approaches to symbolically obtain isoptic curves in GeoGebra in an automated, interactive process. Both methods are based on computing implicit locus equations, by using algebraization of the geometric setup and elimination of the intermediate variables. These methods can be considered as **automatic discovery**.

Our first approach uses **pure computer algebra support of GeoGebra**, utilizing symbolic differentiation.

The second approach hides all details in computer algebra from the user: the input problem is defined by **a purely geometric way**.

In both approaches the output is dynamically changed when using a slider bar or the free points are dragged.

Programming the internal GeoGebra computations is an on-going work with various challenges in optimizing computations and to avoiding unnecessary extra curves in the output.

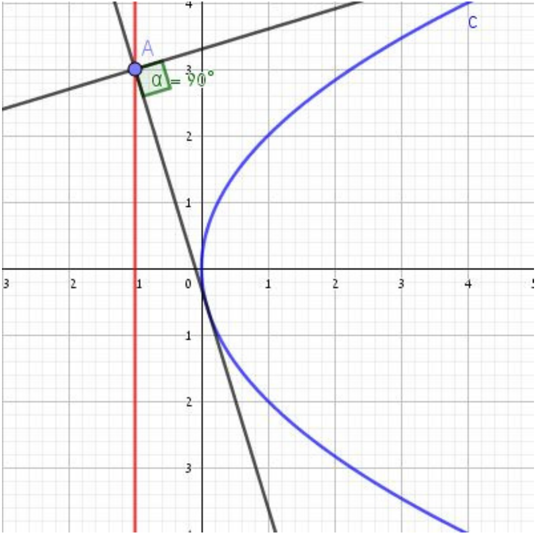
Isoptic curves

Let $\mathcal{C}$ be a plane curve. For a given angle θ such that $0 \leq \theta \leq 180^\circ$, a θ-*isoptic curve* (or simply a θ-isoptic) of $\mathcal{C}$ is the geometric locus of points M through which passes a pair of tangents with an angle of θ between them.
If $\theta = 90^\circ$, i.e. if the tangents are perpendicular, then the isoptic curve is called an *orthoptic* curve.
Isoptic curves may either exist or not, depending on the given curve and on the angle.

Orthoptics of conics

Parabola

The orthoptic curve of a parabola is its *directrix*.
If the parabola has equation $y^2 = 2px$ (for p a non-zero real), then its directrix has equation $x = p/2$.



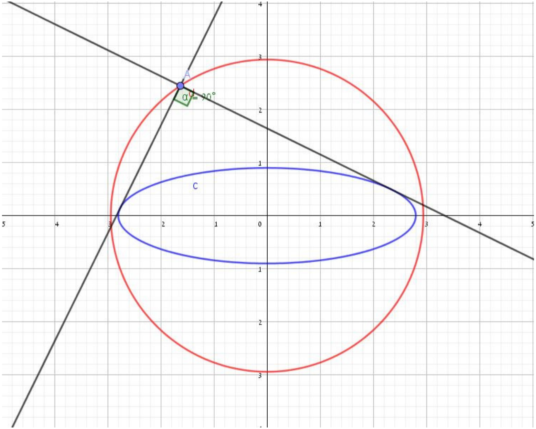
https://www.geogebra.org/m/pwrWy9dG

Orthoptics of conics

Ellipse

The orthoptic curve of an ellipse is its *director circle*.

If the ellipse is given by the canonical equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the director circle has the equation $x^2 + y^2 = a^2 + b^2$.

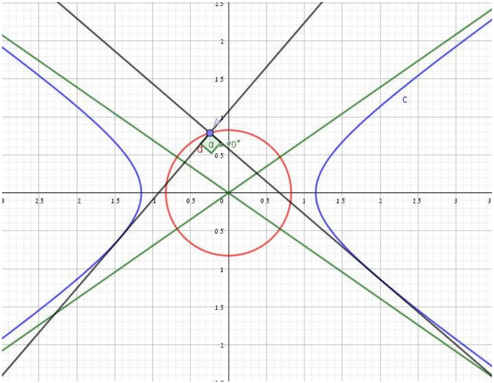


https://www.geogebra.org/m/SkQ5qxYr

Orthoptics of conics

Hyperbola

The existence of an orthoptic curve for a hyperbola depends on the eccentricity c/a, where $c^2 = a^2 - b^2$.
If it exists, the orthoptic curve of the hyperbola with canonical equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (i.e. the focal axis is the x=axis) is the circle whose equation is $x^2 + y^2 = a^2 - b^2$, also called the *director circle*.



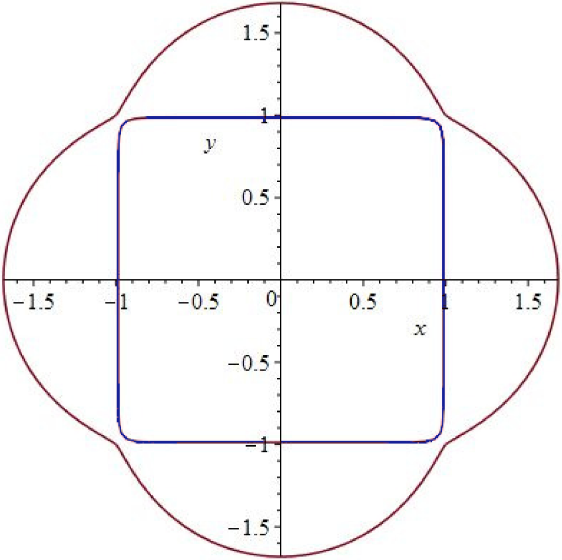
https://www.geogebra.org/m/tZcGGrCm

Previous and related work

- Dana-Picard, Th., Mann, G. and Zehavi, N.: *From conic intersections to toric intersections: the case of the isoptic curves of an ellipse*, The Montana Mathematical Enthusiast 9 (1), pp. 59–76. 2011.
- Dana-Picard, Th.: *An automated study of isoptic curves of an astroid*, Preprint, JCT, 2018.
- Dana-Picard, Th. and Naiman, A.: *Isoptics of Fermat curves*, Preprint, JCT, 2018.
- Miernowski, A. and Mosgawa, W.: *Isoptics of Pairs of Nested Closed Strictly Convex Curves and Crofton-Type Formulas*, Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry 42 (1), pp. 281–288. 2001.
- Szałkowski, D.: *Isoptics of open rosettes*, Annales Universitatis Mariae Curie-Skłodowska, Lublin – Polonia LIX, Section A, pp. 119–128, 2005.
- Csima, G.: *Isoptic curves and surfaces*. PhD thesis, BUTE, Math. Institute, Department of Geometry, Budapest, 2017.

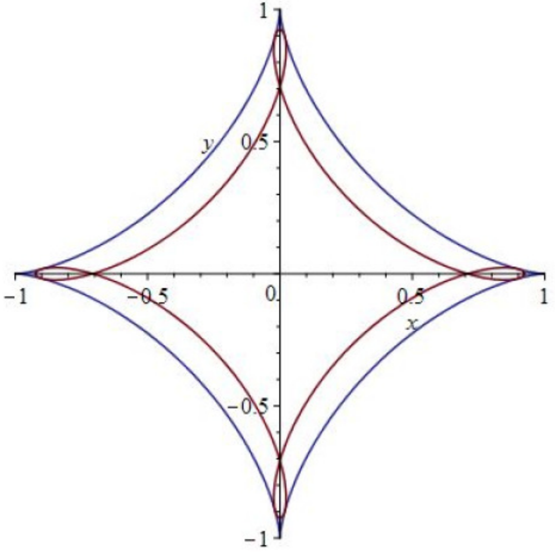
Examples of previous work

The orthoptic of a closed Fermat curve, $x^{16} + y^{16} = 1$



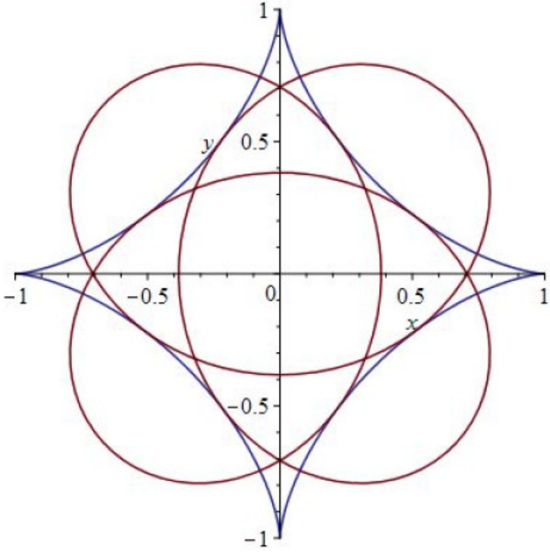
Examples of previous work

45°-isoptic of an astroid, $x^{2/3} + y^{2/3} = 1$



Examples of previous work

$135°$-isoptic of an astroid, $x^{2/3} + y^{2/3} = 1$



Two novel approaches in GeoGebra

An overview

- Both
  - can be considered as **automatic discovery**,
  - deliver an algebraic output: a polynomial (with its graphical representation) via Gröbner bases and elimination.

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  - uses **pure computer algebra support of GeoGebra**: symbolic differentiation of the input formula,
  - allows the output to be changed dynamically with a slider bar (**dynamic study**),
  - can do observations up to quartic curves (due to computational challenges).

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  - allows the output to be changed dynamically with a slider bar (**dynamic study**),
  - can do observations up to quartic curves (due to computational challenges).
- The second approach
  - hides all details in computer algebra from the user: the input problem is given in a **a purely geometric way**,
  - is a handy method for **a new kind of man and machine communication**,
  - works only for certain conics.

The first approach

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3. Compute the partial derivatives $p_{x,A} = F'_x(x_A, y_A)$, $p_{x,B} = F'_x(x_B, y_B)$, $p_{y,A} = F'_y(x_A, y_A)$ and $p_{y,B} = F'_y(x_B, y_B)$.

The first approach
(cont'd)

4. Now, when speaking about orthoptic curves, we can assume that
$$p_{x,A} \cdot p_{x,B} + p_{y,A} \cdot p_{y,B} = 0, \tag{3}$$
otherwise, when speaking about θ-isoptics, the following equation holds:
$$(p_{x,A} \cdot p_{x,B} + p_{y,A} \cdot p_{y,B})^2 = \cos^2\theta \cdot (p_{x,A}^2 + p_{y,A}^2) \cdot (p_{x,B}^2 + p_{y,B}^2). \tag{3'}$$

The first approach

(cont''d)

5. When defining a point $P(x, y)$ that is an element of both tangents t_1 and t_2 to c, the points
   A, $A' = (x_A + p_{y,A}, y_A - p_{x,A})$ and P must be collinear;
   for the same reason, also
   B, $B' = (x_B + p_{y,B}, y_B - p_{x,B})$ and P are collinear.

The first approach
(cont''d)

5. When defining a point $P(x, y)$ that is an element of both tangents t_1 and t_2 to c, the points A, $A' = (x_A + p_{y,A}, y_A - p_{x,A})$ and P must be collinear; for the same reason, also B, $B' = (x_B + p_{y,B}, y_B - p_{x,B})$ and P are collinear. So the following equations hold:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_A + p_{y,A} & y_A - p_{x,A} & 1 \\ x & y & 1 \end{vmatrix} = 0, \tag{4}$$

$$\begin{vmatrix} x_B & y_B & 1 \\ x_B + p_{y,B} & y_B - p_{x,B} & 1 \\ x & y & 1 \end{vmatrix} = 0. \tag{5}$$

The first approach

(cont'''d)

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   This technique ("elimination theory", "automated geometry theorem proving", "automated discovery") is discussed in detail in:
   - Cox, D., Little, J. and O'Shea, D.: *Ideals, varieties and algorithms*. Third edition. Springer, 2007.
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   Theoretically, the obtained implicit equation is a multiple of the algebraic closure of the geometrically expected set. That is, some factors of the obtained implicit equation will contain the expected curve.

Examples

The orthoptic of $y = x^4$

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The equations to consider are as follows:

$$
\begin{align}
x_A^4 - y_A &= 0, \tag{1}\\
x_B^4 - y_B &= 0, \tag{2}\\
4x_A^3 \cdot 4x_B^3 + 1 &= 0, \tag{3}\\
-4x_A^4 + 4x_A^3 x + y_A - y &= 0, \tag{4}\\
-4x_B^4 + 4x_B^3 x + y_B - y &= 0. \tag{5}
\end{align}
$$

Examples

The orthoptic of $y = x^4$ (cont'd)

After eliminating all variables but x and y from this system by using a CAS, we obtain the equation

$$\left(65536x^6 + 196608x^4y^2 + 196608x^2y^4 - 41472x^2y + 65536y^6 + 13824y^3 + 729\right) \cdot$$
$$(16777216x^6y^3 + 50331648x^4y^5 + 5308416x^4y^2 + 50331648x^2y^7 +$$
$$5308416x^2y^4 + 559872x^2y + 16777216y^9 - 1769472y^6 - 186624y^3 + 19683) = 0.$$

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$$f = x^2y + y^3 - 3/8\, y^2\sqrt[3]{2} - \frac{9\, y\sqrt[3]{4}}{64} + \frac{27}{256}.$$

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According to GeoGebra's numerical precision the cubic $f = 0$ is indeed the orthoptic of $y = x^4$.

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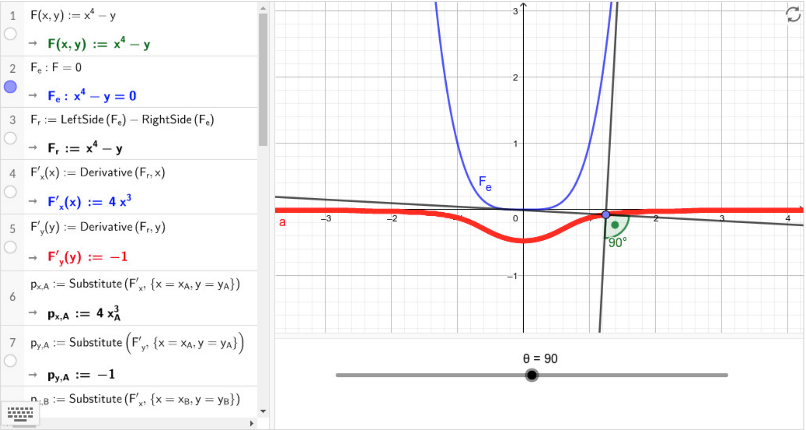
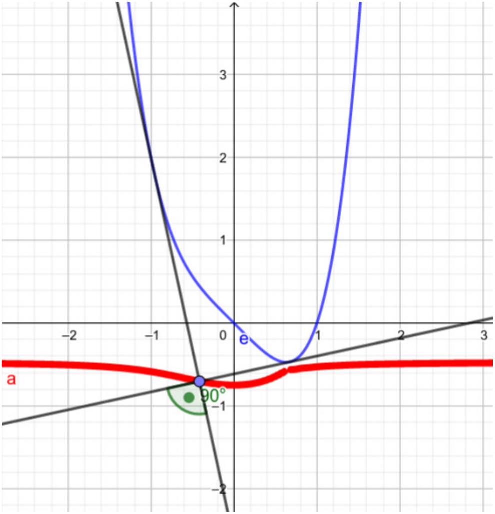
The orthoptic of $y = x^4$ (cont''d)



https://www.geogebra.org/m/JvhNwAzF

Examples

The orthoptic of $y = x^4 - x$



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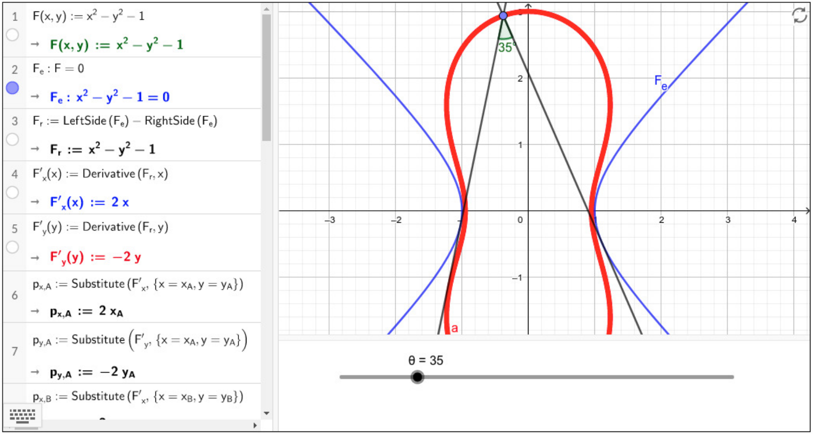
Examples

35°-isoptic of a hyperbola



Examples

35°-isoptic of a hyperbola (cont'd)

Algebraically, after elimination, GeoGebra obtains

$$\begin{aligned}
&2x^{14} - 2y^{14} - c^2x^{12} - c^2y^{12} - 10x^2y^{12} - 18x^4y^{10} - 10x^6y^8 + 10x^8y^6 + 18x^{10}y^4 \\
&+ 10x^{12}y^2 - 6c^2x^2y^{10} - 15c^2x^4y^8 - 20c^2x^6y^6 - 15c^2x^8y^4 - 6c^2x^{10}y^2 - 23x^{12} \\
&- 23y^{12} + 12c^2x^{10} - 12c^2y^{10} - 58x^2y^{10} - 25x^4y^8 + 20x^6y^6 - 25x^8y^4 - 58x^{10}y^2 \\
&- 36c^2x^2y^8 - 24c^2x^4y^6 + 24c^2x^6y^4 + 36c^2x^8y^2 + 112x^{10} - 112y^{10} - 60c^2x^8 \\
&- 60c^2y^8 - 80x^2y^8 + 32x^4y^6 - 32x^6y^4 + 80x^8y^2 - 48c^2x^2y^6 + 24c^2x^4y^4 - 48c^2x^6y^2 \\
&- 300x^8 - 300y^8 + 160c^2x^6 - 160c^2y^6 + 144x^2y^6 - 136x^4y^4 + 144x^6y^2 + 96c^2x^2y^4 \\
&- 96c^2x^4y^2 + 480x^6 - 480y^6 - 240c^2x^4 - 240c^2y^4 + 544x^2y^4 - 544x^4y^2 + 288c^2x^2y^2 \\
&- 464x^4 - 464y^4 + 192c^2x^2 - 192c^2y^2 + 608x^2y^2 - 64c^2 + 256x^2 - 256y^2 - 64 = 0,
\end{aligned}$$

where $c = \cos^2\left(\frac{7}{36}\pi\right)$.

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\end{aligned}
$$

where $c = \cos^2\left(\frac{7}{36}\pi\right)$. After factorization this can be simplified to

$$cx^4 + 2cx^2y^2 + cy^4 - x^4 - 2\,x^2y^2 - 4cx^2 - y^4 + 4cy^2 + 4c = 0,$$

that is, the isoptic curve is a quartic (containing also the set of points for the 145°-isoptic).

Computational features of the first approach

- Fast computations for conics (dragging of θ is possible)
- Feasible (but slow) computations for certain quartics
- Infeasible computations for most quartics and other higher degree polynomials
- GeoGebra's *CAS View* is involved
- In most cases, the output contains additional factors that have no geometrical meaning ("extended output")
- GeoGebra's *Graphics View* correctly plots the extended output
- Factorization of the extended output may be incomplete in GeoGebra (Maple or Singular can be used for absolute factorization): the minimal algebraic form of the curve is difficult to determine

The second approach

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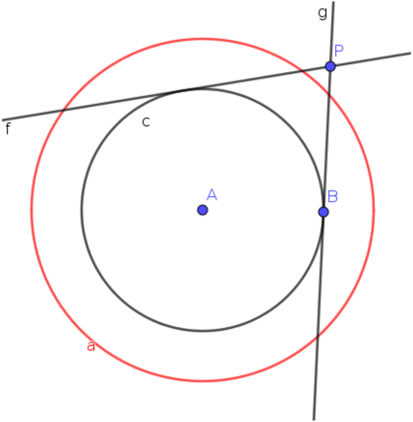
$\texttt{LocusEquation}(f == g, P)$

The second approach

Orthoptic of a circle



https://www.geogebra.org/m/z2uNpHCU

$\texttt{LocusEquation}(f \perp g, P)$

Some features of the second approach

- GeoGebra's *CAS View* is not involved
- Each type of input (circle, parabola, ...) must be separately implemented (=programmed) internally in GeoGebra
- Computations are feasible for orthoptics of circle and parabola (moderately slow dragging of θ is possible)
- To obtain isoptics, the `AreCongruent` command must be used
- Computations are slow for isoptics of circle and parabola
- Isoptic curves may contain extra linear components due to algebraic issues
- Other curves (ellipse, hyperbola and non-conics) are not yet implemented
- The output may contain additional factors that have no geometrical meaning ("extended output")
- Finding the "best" equation system describing the geometric setup can be tricky

Creating the equation system programmatically

GeoGebra's source code is at `https://github.com/geogebra/geogebra`

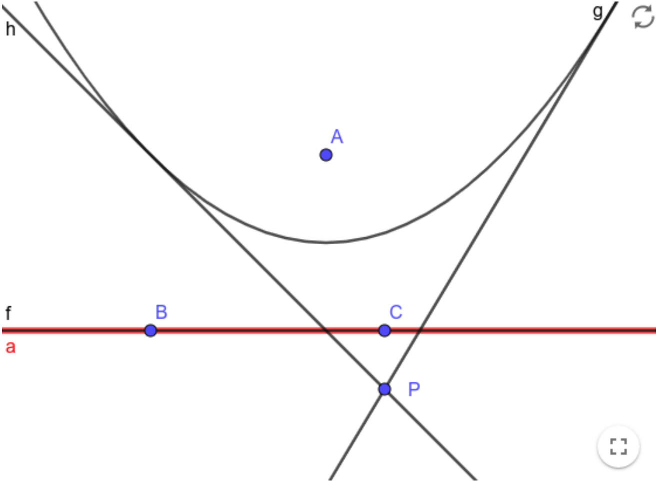
- AlgoTakeString
- AlgoTangentLine
- AlgoTangentLineND
- AlgoTangentPoint
- AlgoTangentPointND
- AlgoText
- AlgoTextCorner
- AlgoTextLength
- AlgoTextToUnicode
- AlgoTransformation
- AlgoTranslate
- AlgoTranslateVector
- AlgoTurningPointPolyInterval
- AlgoTurningPointPolynomial
- AlgoTwoNumFunction
- AlgoUnicodeToLetter
- AlgoUnicodeToText
- AlgoUnitOrthoVectorLine
- AlgoUnitOrthoVectorVector
- AlgoUnitVector
- AlgoUnitVector2D
- AlgoUnitVectorLine
- AlgoUnitVectorVector
- AlgoVector
- AlgoVectorPoint

```java
            }

            PPolynomial[] botanaPolynomialsThis = new PPolynomial[5];

            // coordinates of F'
            PVariable f_1 = new PVariable(kernel);
            PVariable f_2 = new PVariable(kernel);

            // F' is on the directrix (we need to declare it)
            botanaPolynomialsThis[0] = PPolynomial.collinear(f_1, f_2,
                    vparabola[4], vparabola[5], vparabola[6],
                    vparabola[7]);
            // PF' = PF
            botanaPolynomialsThis[1] = PPolynomial.equidistant(f_1, f_2,
                    vPoint[0], vPoint[1], vparabola[8], vparabola[9]);
            // FF' is perpendicular to PT
            botanaPolynomialsThis[2] = PPolynomial.perpendicular(vparabola[8], vparabola[9],
                    f_1, f_2, botanaVarsThis[2], botanaVarsThis[3],
                    botanaVarsThis[0], botanaVarsThis[1]);
            // F'T is perpendicular to the directrix
            botanaPolynomialsThis[3] = PPolynomial.perpendicular(f_1, f_2,
                    botanaVarsThis[0], botanaVarsThis[1], vparabola[4], vparabola[5],
                    vparabola[6], vparabola[7]);
            // T=P is not allowed
            botanaPolynomialsThis[4] = (PPolynomial
                    .sqrDistance(botanaVarsThis[0], botanaVarsThis[1],
                            botanaVarsThis[2], botanaVarsThis[3])
                    .multiply(new PPolynomial(new PVariable(kernel))))
                    .subtract(new PPolynomial( coeff: 1));

            botanaPolynomials.put(geo, botanaPolynomialsThis);
            return botanaPolynomialsThis;
        }

        // Ellipse and hyperbola cannot be distinguished.
        if (c.isEllipse() || c.isHyperbola()) {
```

AlgoTangentPoint › getBotanaPolynomials()

Examples

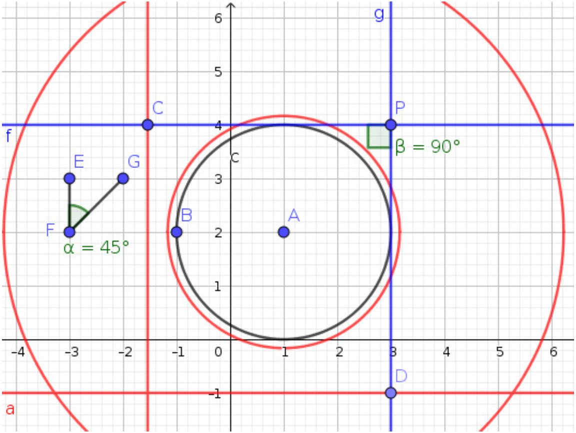
Orthoptic of a parabola



https://www.geogebra.org/m/dtgzjzcj

Examples

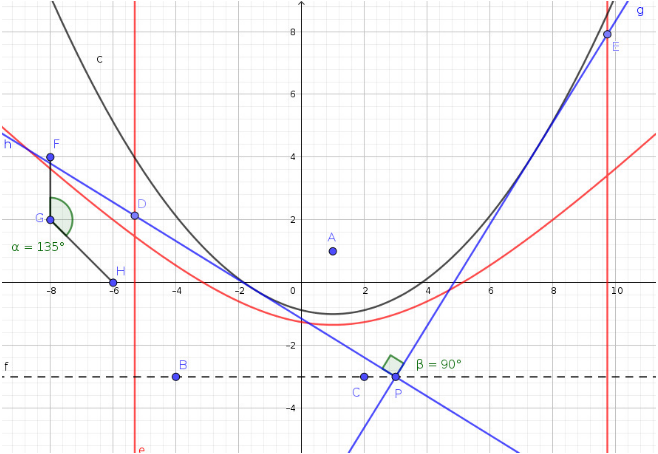
45°-isoptic of the circle



`LocusEquation(AreCongruent(`α`,`β`),`P`)`

Examples

135°-isoptic of the parabola



LocusEquation(AreCongruent(α,β),P)

Conclusion

- No longer a researchers-only topic? Students can be involved!
- Another application of Gröbner bases and elimination (for polynomial input)
- Experiments exploiting (computer) algebraic and (dynamic geometric) graphical representations
- Further studies may involve more efficient computations and further tricks

Thank you for your kind attention!

Thank you for your kind attention!